

Sliding Mode Control of a Thermal Mixing Process

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Abstract

In this paper we consider the robust control of a thermal mixer using multivariable Sliding Mode Control (SMC). The mixer consists of a mixing chamber, hot and cold fluid valves, and an exit valve. The commanded positions of the three valves are the available control inputs, while the controlled variables are total mass flow rate, chamber pressure and the density of the mixture inside the chamber. Unsteady thermodynamics and linear valve models are used in deriving a 5th order nonlinear system with three inputs and three outputs. An SMC controller is designed to achieve robust output tracking in the presence of unknown energy losses between the chamber and the environment. The usefulness of the technique is illustrated with a simulation.

1 Introduction

The problem of dosing the amounts of hot and cold fluid to produce a mixture having a desired temperature and flow rate is familiar to everyone, including people who are not control experts. In addition to the mundane shower example, the problem appears in industrial practice as well [6, 9, 1]. In this paper we analyze the problem of obtaining the desired exit flow rate and temperature, while at the same time governing the pressure of the mixture at a certain location. Of course, the inclusion of a third independent quantity to be controlled requires the addition of a third control variable, in this case a valve controlling the flow of mixture. Fig. 1 depicts the set-up. A mathematical model for the mixer in the context of propellant delivery in rocket engine test stands has been developed, and some controllers have been designed. In particular, small-signal linearization [1] and feedback linearization [9] controllers have been developed. In this paper we expand the model to include first-order, linear dynamics for the valves and incorporate the effect of an unknown energy disturbance. In particular, we consider the disturbance to be the heat transfer between the chamber and the environment. This quantity is indeed difficult to measure and has practical relevance. Although possible to model in principle, a quantitative description

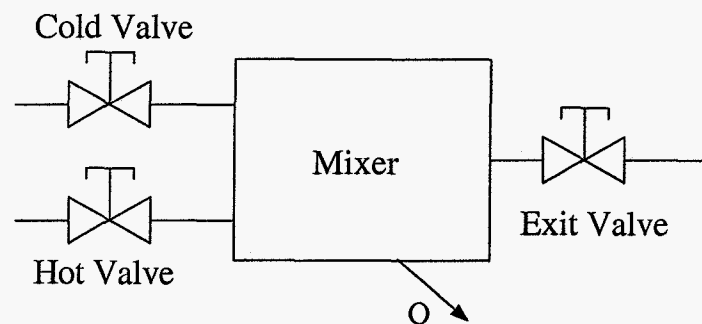


Figure 1: Mixer Subsystem

of the heat transfer through mixer walls involves awkward convection correlations with parameters which are heavily dependent on temperature and flow characteristics. The convection coefficients can be extremely uncertain and often have to be found through iteration, even for the steady heat transfer calculations. For control purposes, it is far more convenient to treat the heat transfer (and other unmodeled energy losses) as a lumped and unknown bounded disturbance. A bound on the external heat transfer is relatively easy to find. Other energy disturbances that intervene in the energy conservation equations can also be included to create a "bundled" uncertainty to be considered when tuning the controller. In order to simplify calculations, we assume that both cold and hot fluids are the same ideal gas obeying the well-known formula $P = \rho RT$, where P is the pressure, ρ is the density, T is the temperature and R is a constant particular to a given gas. As in [6, 9, 8], it is assumed that the valves conserve enthalpy and that source fluid properties, as well as exit pressure, remain constant. Although the temperature of the mixture past the exit valve is a controlled variable, it will be substituted by mixture density inside the chamber to avoid cumbersome formulas. It will be assumed that the reference pressure and density are chosen so that the desired exit temperature is obtained. This is possible since a given mixer pressure and density, along with exit pressure, uniquely determine an exit temperature through thermodynamic property data. Sliding Mode Control (SMC) is especially suited for use in problems such as the one at hand, due to its insensitivity to certain kinds of disturbances and parameter variations [3, 14, 13] and, in our case, the achievement of output decoupling, which facilitates tuning. Applications of SMC to thermal systems

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have been reported, for instance, in [5, 11, 4, 12, 2].

2 Mathematical Model

As explained in [6, 9, 8], the transient versions of the mass and energy conservation equations are used in deriving the two basic model equations. It is convenient to write the equations in term of a pressure state instead of internal energy, since the flow rates can be directly computed from the state variables without further reference to thermodynamic relations. For the ideal gas, the following identities and definitions are in order

$$h = K_p T = \frac{PK_p}{\rho R} \quad (1)$$

$$u = K_v T = \frac{PK_v}{\rho R} \quad (2)$$

$$k = \frac{K_p}{K_v} \quad (3)$$

$$R = K_p - K_v \quad (4)$$

where h is the enthalpy, u is the internal energy, K_p and K_v are the constant pressure and constant volume heat capacities, respectively, and k is the heat capacity ratio K_p/K_v . As done in [7], we define a *valve flow function* to be a function of three variables satisfying

$$F(P, \rho, \zeta) = \zeta f(P, \rho)$$

Thus, the mass flow rates through the two inlet valves and the exit valve are given by

$$F_1(P, \rho, C_{vc}) = C_{vc} f_1(P, \rho) \text{ for the cold fluid (5)}$$

$$F_2(P, \rho, C_{vh}) = C_{vh} f_2(P, \rho) \text{ for the hot fluid (6)}$$

$$F_e(P, \rho, C_{ve}) = C_{ve} f_e(P, \rho) \text{ for the exit fluid (7)}$$

where P and ρ are the pressure and density of the gas inside the chamber, and C_{vc} , C_{vh} and C_{ve} are valve flow coefficients. Note that the input flows depend also on the thermodynamic properties of the source gases, regarded as constant parameters in Eqs. (5,6). The exit flow depends also on the pressure P_s at the outlet of the exit valve, which is again considered as a constant parameter in Eq. (7). The flow formulas have a fairly general form, with the only requirement that they be linear in the valve position. We assume that the valves behave as a first order linear system between commanded position and flow coefficient. Valve dynamics are expressed as

$$\dot{C}_v = A_c C_v + B_c \zeta \quad (8)$$

where $A_c = \text{diag}(-1/\tau_1, -1/\tau_2, -1/\tau_3)$, $B_c = [\beta_1 \ \beta_2 \ \beta_3]^T$, τ_i are the time constants of the cold, hot and exit valves and β_i are their respective gains; $\zeta = [\zeta_1 \ \zeta_2 \ \zeta_3]^T$ is the vector of control inputs to the

cold, hot and exit valves, and $C_v = [C_{vc} \ C_{vh} \ C_{ve}]^T$ is the vector of valve flow coefficients. Using the above definitions and identities, the final form of the ideal gas mixer model with heat transfer disturbance can be written as

$$\dot{\rho} = \frac{1}{V} C_v^T f \quad (9)$$

$$\dot{P} = \frac{1}{V} C_v^T A_p f + \frac{R}{V K_v} Q \quad (10)$$

$$\dot{C}_v = A_c C_v + B_c \zeta \quad (11)$$

$$y = [\rho \ u \ C_{ve} f_e]^T \quad (12)$$

with A_p given by Eq. (13).

$$A_p = \text{diag} \left[\frac{R h_1}{K_v}, \frac{R h_2}{K_v}, \frac{k P}{\rho} \right] \quad (13)$$

and $f = [f_1 \ f_2 \ -f_e]^T$. Q represents the rate of heat transfer (i.e., expressed in Watts in the SI system) between mixer walls and the environment.

3 Sliding Mode Control Design

As seen in Eq. (10), the unknown heat transfer Q appears in the equations as a disturbance to which the system can be made insensitive through SMC. Pressure losses due to friction can be combined with the heat transfer to create a bundled disturbance. For simplicity we choose three independent linear sliding manifolds, although other choices are possible [7]. Let the sliding manifolds be

$$s_1 = \dot{\rho}_d - \dot{\rho} + c_\rho(\rho_d - \rho) \quad (14)$$

$$s_2 = \dot{P}_d - \dot{P} + c_P(P_d - P) \quad (15)$$

$$s_3 = c_w(w_d - C_{ve} f_e) \quad (16)$$

where ρ_d , P_d and w_d denote the desired trajectories for the density, pressure and exit mass flow outputs, respectively, and c_ρ , c_P and c_w are sliding coefficients. Note that when the system is in the sliding regime, that is, $\dot{s}_i = 0$, $i = 1, 2, 3$, it is of degree five, the same as the uncontrolled plant. Therefore there is no zero dynamics with respect to the sliding manifolds and the system is trivially minimum-phase. Having established the stability of the zero dynamics, it is now possible to follow the steps of a standard MIMO SMC design. The first step is to write the 5th-order system in terms of output derivatives. Differentiating the first output twice leads to

$$\ddot{y}_1 = \frac{C_v^T}{V} \left(A_c f + \frac{df}{dt} \right) + \frac{\zeta^T B_c f}{V}$$

where

$$\frac{df}{dt} = \left[\frac{df_1}{dt} \ \frac{df_2}{dt} \ -\frac{df_e}{dt} \right]^T$$

Similarly, differentiating the second output twice gives

$$\ddot{y}_2 = \frac{C_v^T}{V} \left[\left(A_c A_p + \frac{dA_p}{dt} \right) f + A_p \frac{df}{dt} \right] + \frac{\zeta^T B_c A_p f}{V} + \frac{R}{V K_v} \dot{Q}$$

The third output, however, needs to be differentiated only once for the control to appear:

$$\dot{y}_3 = C_{ve} \frac{df_e}{dt} + \left(-\frac{1}{\tau_3} C_{ve} + \beta_3 \zeta_3 \right) f_e$$

It is desired that s_1 , s_2 and s_3 reach zero after some finite time and remain zero despite of the heat transfer disturbance. Choosing Lyapunov functions

$$V_i = \frac{1}{2} s_i^2$$

for $i = 1, 2, 3$, it is sufficient to ensure that $\dot{V}_i < 0$ at all times to guarantee the convergence of s_i . To reach zero in finite time, a commonly used choice is to enforce

$$\dot{V}_i = s_i \dot{s}_i \leq -\eta_i |s_i| \quad (17)$$

It can be shown [10] that the above implies that s_i reaches zero in a time t_i given by

$$t_i \leq \frac{|s_i(t=0)|}{\eta_i}$$

Enforcement of Eq. (17) can be achieved by using the equality for s_1 and s_3 , since the heat transfer uncertainty is not involved in the expressions. Thus, let $s_i \dot{s}_i = -\eta_i |s_i|$ for $i = 1, 2$. This is equivalent to $\dot{s}_i = -\eta_i \text{sign}(s_i)$. Equality cannot be guaranteed for s_2 , since this would require knowledge of Q and \dot{Q} . The SMC will rely, however, on the knowledge of a bound for these quantities and on the ability to increase the sliding gain η_2 to counteract the disturbance. Enforcing $\dot{s}_i = -\eta_i \text{sign}(s_i)$ for $i = 1, 3$ leads to

$$\frac{\zeta^T B_c f}{V} = \Gamma_1 \quad (18)$$

$$\zeta^T G f = \Gamma_3 \quad (19)$$

where

$$\Gamma_1 = \ddot{p}_d + c_p \dot{p}_d + \eta_1 \text{sign}(s_1) - \frac{C_v^T}{V} \left((A_c + c_p I) f + \frac{df}{dt} \right)$$

$$\Gamma_3 = c_w \left(\dot{w}_d + \frac{1}{\tau_3} C_{ve} f_e - C_{ve} \frac{df_e}{dt} \right) + \eta_3 \text{sign}(s_3)$$

and G is a 3-by-3 matrix with zeros in every entry, except for $G(3, 3) = -c_w \beta_3$. For the remaining sliding variable, we choose

$$\frac{\zeta^T B_c A_p f}{V} = \Gamma_2 \quad (20)$$

where

$$\Gamma_2 = \ddot{p}_d + c_p \dot{p}_d + \eta_2 \text{sign}(s_2) - \frac{C_v^T}{V} \left[\left((A_c + c_p I) A_p + \frac{dA_p}{dt} \right) f + A_p \frac{df}{dt} \right]$$

so that

$$\dot{s}_2 = -\eta_2 \text{sign}(s_2) - \frac{R \dot{Q}}{V K_v} - \frac{k R}{V} Q$$

We want

$$s_2 \dot{s}_2 \leq -\eta'_2 |s_2| \quad (21)$$

with $\eta'_2 > 0$. Let

$$\eta_2 = \eta'_2 + \frac{R \bar{Q}}{V K_v}$$

where \bar{Q} is a known bound such that

$$|\dot{Q} + c_p Q| < \bar{Q}$$

at all times. Then it is straightforward to show that Eq. (21) is satisfied. Since $\eta'_2 > 0$, it is then sufficient to choose

$$\eta_2 > \frac{R \bar{Q}}{V K_v} \quad (22)$$

The equations that yield the control input, Eqs. (18), (20) and (19) can be compactly expressed in matrix form as a system of equations linear in ζ :

$$\begin{bmatrix} \frac{f^T B_c}{V} \\ \frac{f^T A_p B_c}{V} \\ f^T G \end{bmatrix} \zeta = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \end{bmatrix} \quad (23)$$

The existence of singularities preventing the inversion of the above system matrix and the more general question of state trajectory and control admissibility is currently being studied by the authors in the broader context of constrained SMC. For constant output setpoints, however, it is possible to find conditions on the desired density, pressure and exit flow so that the steady state values of the valve flow coefficients are positive. The conditions reduce to [6]

$$h_c < \bar{h} < h_h$$

where \bar{h} is the enthalpy of the mixture that corresponds to the steady pressure and density setpoints, and h_c and h_h are the enthalpies of the cold and hot gases, respectively. Interestingly, this basic admissibility result is reduced to an expression involving only enthalpies. Note that the invertibility of the decoupling matrix for the feedback linearization design of [9] likewise reduces to $h_c \neq h_h$.

Parameter	Value	Units
P_1	6.894	MPa
T_1	311.1	$^{\circ}K$
P_2	5.516	MPa
T_2	422.22	$^{\circ}K$
P_s	206.842	kPa
R	296.8	J/kg $^{\circ}K$
K_p	1041.6	J/kg $^{\circ}K$
k	1.398	n.a
P_0	3.447	MPa
ρ_0	48.05	kg/m 3
$C_{v1,2,3}$	10	n.a
$\tau_{1,2,3}$	0.1	sec
β_1	1	n.a
β_2	2	n.a
β_3	3	n.a
V	0.07079	m 3

Table 1: System Parameters for Sliding Mode Controller Simulation

4 Simulation Example

In this example, we choose the SI system of units and nitrogen as the working gas. The following design specifications are used:

- Obtain a steady exit flow of 10 kg/s with a mixer operating pressure of 4 MPa.
- The temperature of the exit flow must be $337^{\circ}K = 64^{\circ}C$.
- The heat transfer disturbance is $Q = 10\sin(10t)$.
- The steady operating point must be reached in less than 25 seconds, for known initial conditions given in Table 1.
- Valve chattering must not be observed.

Other system parameters are summarized in Table 1.

SMC Tuning The values of w_d and P_d are directly taken from the design specifications. It is necessary to calculate the value of ρ_d which will result in the desired exit temperature. For this, we have that the isenthalpic process at the exit valve is also isothermal, since it is assumed that K_p is constant. Therefore we choose $\rho_d = \frac{P_d}{RT_d} = 40\text{kg/m}^3$. In order to meet the settling time requirement, we choose $c_p = c_P = c_w = 100$, $\eta_1 = \eta_3 = 120$ and $\eta_2 = 2.4062 \times 10^6$. With the known initial conditions and setting $\dot{\rho}_d = \ddot{\rho}_d = \dot{P}_d =$

$\ddot{P}_d = \dot{w}_d = 0$ we obtain the reaching times:

$$\begin{aligned} t_{r1} &\leq \frac{|s_1(t=0)|}{\eta_1} = 7.08 \text{ sec} \\ t_{r2} &\leq \frac{|s_2(t=0)|}{\eta_2} = 19.5 \text{ sec} \\ t_{r3} &\leq \frac{|s_3(t=0)|}{\eta_3} = 5.83 \text{ sec} \end{aligned}$$

The overall settling time is the sliding surface reaching time plus the settling time during sliding, given by $t_{s1} = t_{s2} = t_{s3} = 4/100 = 0.04\text{sec}$. The above gain choices thus satisfy the settling time requirements. Finally, we check that the controller can attain its objective despite the presence of the heat transfer disturbance. The value of \bar{Q} is computed as

$$\bar{Q} = |\dot{Q} + c_P Q| = 1100$$

We see that the selection of η_2 is appropriate, since

$$\eta_2 = 2.4062 \times 10^6 > \frac{\bar{Q}R}{VK_v}$$

To avoid chattering, the signum functions present in the control law are replaced by finite slope approximations consisting of saturation functions:

$$\text{sign}(s_i) \approx \text{sat}(s_i/\phi_i)$$

where

$$\text{sat}(x) = \begin{cases} x & \text{if } |x| \leq 1 \\ 1 & \text{if } x > 1 \\ -1 & \text{otherwise} \end{cases} \quad (24)$$

For the simulation example, the values $\phi_1 = \phi_3 = 10$ and $\phi_2 = 1 \times 10^4$ were chosen. Figures 3, 4 and 2 show the simulation results. As it can be seen, the control objectives are satisfied. The second valve position is unacceptable, since it contains a time interval when the valve flow coefficient is negative. Avoidance of reverse flows and negative valve positions were not specified in the control problem. This simulation motivates further study into admissible state and control trajectories of systems under SMC.

5 Conclusions

The paper illustrates how MIMO SMC can be applied to the control of a thermal mixer in the presence of an unknown energy disturbance. The obtained controller guarantees robust tracking of the outputs as long as a decoupling matrix can be inverted. Invertibility of the decoupling matrix is related to a broader problem of constrained control and is currently under investigation.

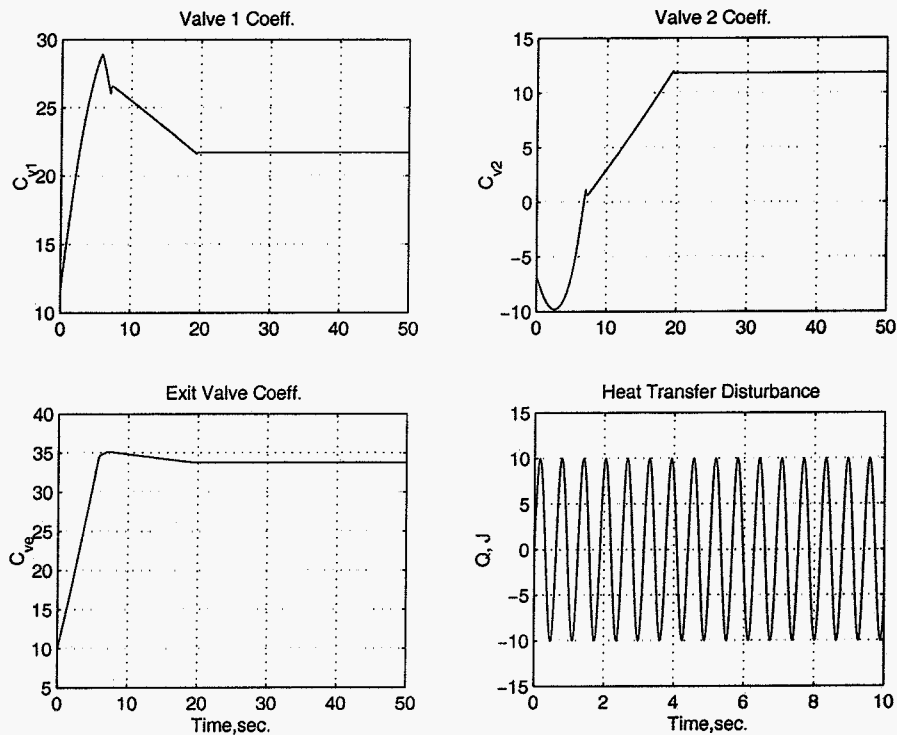


Figure 2: Simulation of Sliding Mode Controller - Control Variables and Disturbance

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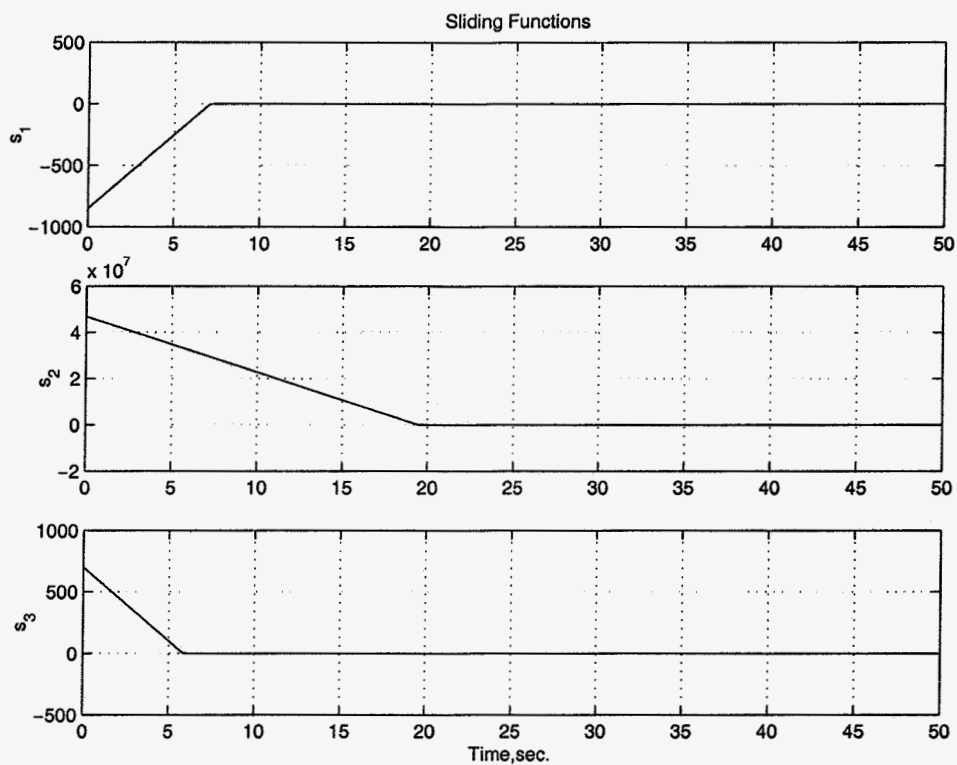


Figure 3: Simulation of Sliding Mode Controller - Sliding Variables

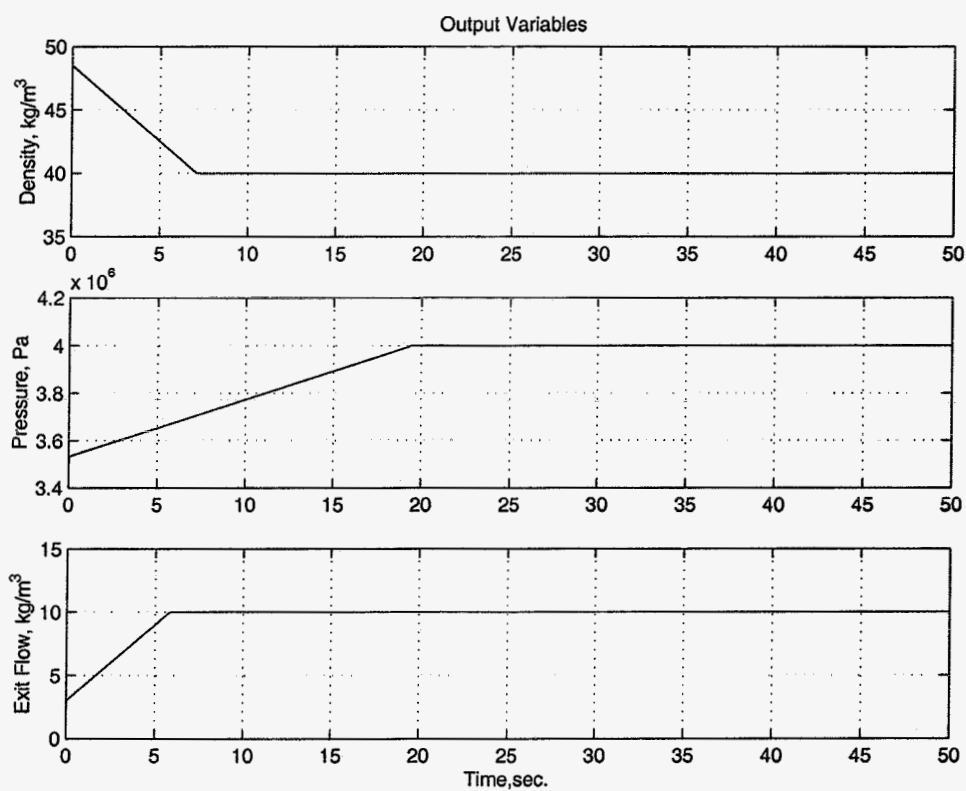


Figure 4: Simulation of Sliding Mode Controller - Output Variables

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